1. The curve with equation $y = f(x)$ where

$$f(x) = x^2 + \ln(2x^2 - 4x + 5)$$

has a single turning point at $x = \alpha$

(a) Show that α is a solution of the equation

$$2x^3 - 4x^2 + 7x - 2 = 0$$

(4)

The iterative formula

$$x_{n+1} = \frac{1}{7}(2 + 4x_n^2 - 2x_n^3)$$

is used to find an approximate value for α.

Starting with $x_1 = 0.3$

(b) calculate, giving each answer to 4 decimal places,

(i) the value of x_2

(ii) the value of x_4

(3)

Using a suitable interval and a suitable function that should be stated,

(c) show that α is 0.341 to 3 decimal places.

(2)

a) $f(x) = x^2 + \ln(2x^2 - 4x + 5)$

$f'(x) = 2x + \dfrac{4x - 4}{2x^2 - 4x + 5}$ ②

turning point has $f'(x) = 0$

$2x + \dfrac{4x - 4}{2x^2 - 4x + 5} = 0$ $\times(2x^2 - 4x + 5)$

$2x(2x^2 - 4x + 5) + 4x - 4 = 0$ ①

$4x^3 - 8x^2 + 14x - 4 = 0$

$2x^3 - 4x^2 + 7x - 2 = 0$ ①

b) $x_{n+1} = \frac{1}{7}(2 + 4x_n^2 - 2x_n^3)$

$x_1 = 0.3$

$x_2 = \frac{1}{7}(2 + 4(0.3)^2 - 2(0.3)^3)$ 1

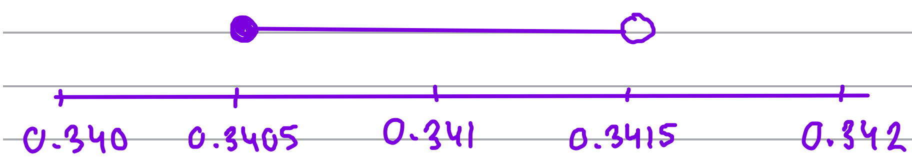
(i) $x_2 = 0.329428\ldots$
$= 0.3294$ (4dp) 1

(ii) $x_4 = 0.339823\ldots$
$= 0.3398$ (4dp) 1

c) let $h(x) = 2x^3 - 4x^2 + 7x - 2$
$h(x) = 0$ represents a turning point of $f(x)$.
want to show that $\alpha = 0.341$ to 3dp.

$h(0.3415) = 0.00366\ldots > 0$
$h(0.3405) = -0.00130\ldots < 0$ 1

- since there is a change in sign
- and $f'(x)$ is a continuous function
- $\alpha = 0.341$ to 3dp 1



2.

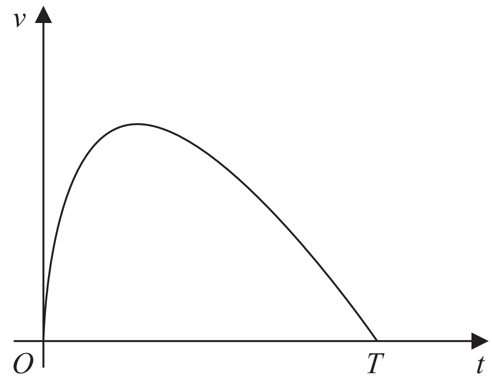


Figure 2

A car stops at two sets of traffic lights.

Figure 2 shows a graph of the speed of the car, $v\,\mathrm{ms}^{-1}$, as it travels between the two sets of traffic lights.

The car takes T seconds to travel between the two sets of traffic lights.

The speed of the car is modelled by the equation

$$v = (10 - 0.4t)\ln(t + 1) \qquad 0 \leqslant t \leqslant T$$

where t seconds is the time after the car leaves the first set of traffic lights.

According to the model,

(a) find the value of T

(1)

(b) show that the maximum speed of the car occurs when

$$t = \frac{26}{1 + \ln(t + 1)} - 1$$

(4)

Using the iteration formula

$$t_{n+1} = \frac{26}{1 + \ln(t_n + 1)} - 1$$

with $t_1 = 7$

(c) (i) find the value of t_3 to 3 decimal places,

(ii) find, by repeated iteration, the time taken for the car to reach maximum speed.

(3)

(a) $(10-0.4t)\ln(t+1)=0$ ← $v=0$ when $t=0$ and when $t=T$.

$10\ln(t+1)-0.4t\ln(t+1)=0$

$+0.4t\ln(t+1)$

$10\ln(t+1)=0.4t\ln(t+1)$

$\div\ln(t+1)$ this is okay because we know $v=0$ when $t=0$, so $T>0$. Then $T+1>0$, so $\ln(t+1)\neq 0$.

$10=0.4t$

$25=t$

$\therefore T=25$ (1)

(b) $v=(10-0.4t)\ln(t+1)$

let $v=f(t)g(t)$

then $v'=f(t)g'(t)+f'(t)g(t)$

$f(t)=10-0.4t \qquad f'(t)=-0.4$

$g(t)=\ln(t+1) \qquad g'(t)=\frac{1}{t+1}$

$\frac{dv}{dt}=\ln(t+1)\times -0.4+(10-0.4t)\times\frac{1}{t+1}$ (2)

$0=-0.4\ln(t+1)+\frac{10-0.4t}{t+1}$ (1) ← max speed when gradient is 0 (at turning point)

$\frac{10-0.4t}{t+1}=0.4\ln(t+1)$

$\times(t+1)$

$10-0.4t=0.4\ln(t+1)\times(t+1)$

$+0.4t$

$10=0.4t\ln(t+1)+0.4\ln(t+1)+0.4t$

$\div 0.4$

$25=t\ln(t+1)+\ln(t+1)+t$

factorise

$25=t(\ln(t+1)+1)+\ln(t+1)$

$-\ln(t+1)$

$25-\ln(t+1)=t(\ln(t+1)+1)$

$\div(1+\ln(t+1))$

$\frac{25-\ln(t+1)}{1+\ln(t+1)}=t$

$$\frac{26}{1+\ln(t+1)} - 1 = t \quad 1$$

(c) $$t_{n+1} = \frac{26}{1+\ln(t_n+1)} - 1$$

$$t_1 = 7$$

$$t_2 = \frac{26}{1+\ln(7+1)} - 1 \; 1 = 7.298 \; 1$$

$$t_3 = \frac{26}{1+\ln(7.298+1)+1} = 7\ 33 \quad 1$$

$t_3 = 7.33$ seconds

3. A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{7x\mathrm{e}^{x}}{\sqrt{\mathrm{e}^{3x} - 2}} \qquad x > \ln \sqrt[3]{2}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{7\mathrm{e}^{x}(\mathrm{e}^{3x}(2 - x) + Ax + B)}{2(\mathrm{e}^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2\mathrm{e}^{3x} - 4}{\mathrm{e}^{3x} + 4}$$

(2)

The equation $x = \dfrac{2\mathrm{e}^{3x} - 4}{\mathrm{e}^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2\mathrm{e}^{3x_n} - 4}{\mathrm{e}^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \dfrac{2\mathrm{e}^{3x} - 4}{\mathrm{e}^{3x} + 4}$ and part of the line with equation $y = x$

Using Diagram 1 on page 42

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

Using a suitable interval and a suitable function that should be stated

(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)

a) $f(x) = \dfrac{7xe^{x}}{(e^{3x}-2)^{1/2}}$

$\dfrac{d}{dx}(7xe^{x})$: let $u = 7x$ $\quad v = e^{x}$

$\dfrac{du}{dx} = 7 \quad \dfrac{dv}{dx} = e^{x}$

1 1

$\dfrac{du}{dx}v + \dfrac{dv}{dx}u = 7e^{x} + 7xe^{x}$

$\dfrac{d}{dx}\left((e^{3x}-2)^{1/2}\right) = \dfrac{1}{2} \times 3e^{3x} \times (e^{3x}-2)^{-1/2}$ 1

$= \dfrac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}$

$f(x) = \dfrac{7xe^{x}}{(e^{3x}-2)^{1/2}}$ using quotient rule

1

$f'(x) = \dfrac{(e^{3x}-2)^{1/2}(7e^{x}+7xe^{x}) - 7xe^{x}\left(\frac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}\right)}{e^{3x}-2}$

$= \dfrac{7(e^{3x}-2)^{-1/2}\left[e^{x}(e^{3x}-2)(1+x) - \frac{3}{2}xe^{x}e^{3x}\right]}{e^{3x}-2}$

factoring out $7(e^{3x}-2)^{-1/2}$

$= \dfrac{7e^{x}\left[(e^{3x}-2)(1+x) - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$

factoring out e^{x}

moving $(e^{3x}-2)^{-1/2}$ to the denominator

Only use the copy of Diagram 1 if you need to redraw your answer to part (c).

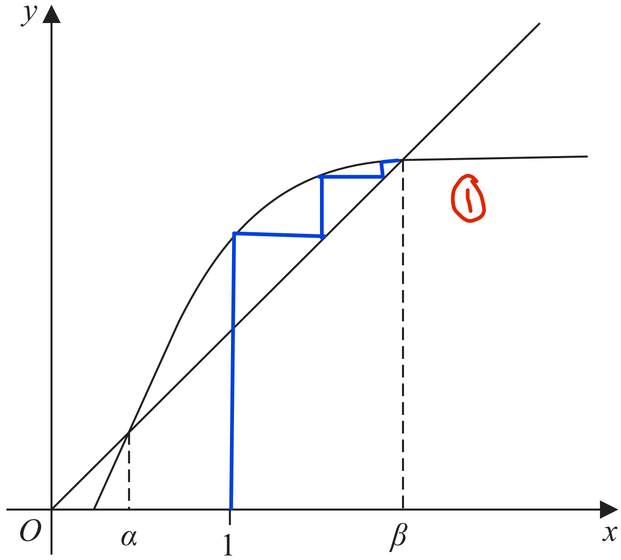
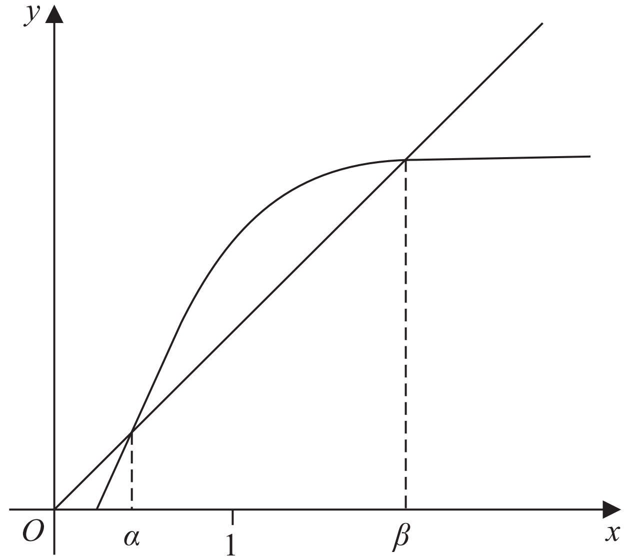


Diagram 1



copy of Diagram 1

expanding bracket

$$f'(x) = \frac{7e^{x}\left[e^{3x} + xe^{3x} - 2 - 2x - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$$

collecting like terms

$$= \frac{7e^{x}\left[e^{3x} - \frac{1}{2}xe^{3x} - 2x - 2\right]}{(e^{3x}-2)^{3/2}}$$

multiplying top and bottom by 2

$$= \frac{7e^{x}(2e^{3x} - xe^{3x} - 4x - 4)}{2(e^{3x}-2)^{3/2}}$$

1

$$= \frac{7e^{x}(e^{3x}(2-x) - 4x - 4)}{2(e^{3x}-2)^{3/2}}$$ as required.

b) turning points have $f'(x)=0$

$$\Rightarrow \frac{7e^{x}\left(e^{3x}(2-x)-4x-4\right)}{2(e^{3x}-2)^{3/2}}=0$$

$e^{x}\neq 0$, multiply by $2(e^{3x}-2)^{3/2}$

$$e^{3x}(2-x)-4x-4=0$$

$$2e^{3x}-xe^{3x}-4x-4=0$$

$$x(e^{3x}+4)=2e^{3x}-4 \quad (1)$$

$$x=\frac{2e^{3x}-4}{e^{3x}+4} \quad (1)$$

c) drawn on diagram

d) $$x_{n+1}=\frac{2e^{3x_n}-4}{e^{3x_n}+4}$$

(i) $x_2=\dfrac{2e^{3}-4}{e^{3}+4}=1.50177\ldots$ (1)

$=1.502$ (3dp) (1)

(ii) $\beta=1.96757\ldots$

$=1.968$ (3dp) (1)

e) α is a solution of $x=\dfrac{2e^{3x}-4}{e^{3x}+4}$

$\therefore$ a solution of $\dfrac{2e^{3x}-4}{e^{3x}+4}-x=0$

so define $h(x) = \frac{2e^{3x}-4}{e^{3x}+4} - x$, so $h(\alpha)=0$.

$h(0.4315) = -0.000297\ldots < 0$
$h(0.4325) = 0.000947 > 0$ (1)

- since there is a change of sign
- and $h(x)$ is continuous
- $\alpha = 0.432$ (to 3dp) (1)